Representations of finite sets and correspondences

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In this joint work with Jacques Thévenaz (EPFL), we consider the category $\mathcal{C}$ of finite sets, where morphisms are given by correspondences instead of maps. We study the abelian category $\mathcal{F}_k$ of *correspondence functors*, i.e. functors from $\mathcal{C}$ to the category of k-modules, where k is a commutative ring. Part of this study relies on the description of some related algebras, and conversely, it leads to new results on these algebras.

• We first consider the k-algebra $\mathcal{E}_E$ of *essential relations* on a finite set E, i.e. the quotient of the algebra $\mathcal{R}_E$ of the monoid of all relations on E by the ideal of relations which can be factorized through a set E' with $|E'| < |E|$. We introduce a nilpotent two sided ideal $\mathcal{N}_E$ of $\mathcal{E}_E$, and show that the quotient algebra $\mathcal{E}_E/\mathcal{N}_E$ is isomorphic to a product of matrix algebras over group algebras indexed by the *order relations* on E. This gives a description of the *simple $\mathcal{E}_E$-modules*: when k is a field, they are parametrized by pairs (R, V) consisting of an order relation R on E, and a simple k-linear representation of the automorphism group $\mathrm{Aut}(E, R)$.

• We next consider some general properties of the category $\mathcal{F}_k$. We show in particular that when k is a field, the *finitely generated* functors are the functors of *finite length*. They are characterized by the *exponential growth* of the dimension of their evaluations. The algebra $\mathcal{R}_E$ of all relations on a finite set E is always *symmetric*, and similarly, we show that a finitely generated correspondence functor is projective if and only if it is injective.

More generally when k is noetherian, any subfunctor of a finitely generated functor is itself finitely generated. We also obtain *stabilization results*, showing that the extension groups between two functors M and N, where M has *bounded type*, can be computed as the extension groups as $\mathcal{R}_E$-modules of their evaluations at E, whenever E is a large enough finite set.

• Finally, we build a *fully faithful* functor $T \to F_T$ from a suitable k-linear category $k\mathcal{L}$ of finite lattices to $\mathcal{F}_k$. This functor is compatible with duality, i.e. it maps the opposite lattice T^{op} to the dual functor $(F_T)^\natural$. We also show that F_T is projective if and only if T is distributive.

The simple correspondence functors over a field k are parametrized by triples (E, R, V) consisting of a finite set E, an order relation R on E, and a simple $k\mathrm{Aut}(E, R)$-module V. To each finite poset (E, R), we attach a fundamental functor $\mathbb{S}_{E,R}$, which allows for a complete description of all simple functors $S_{E,R,V}$. In particular, we obtain a closed formula for the dimension of the evaluation $S_{E,R,V}(X)$ at a finite set X. A consequence of this is a description of all simple $\mathcal{R}_E$-modules, and a closed formula for their dimension.